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Solutions of the Lorentz–Dirac equation in the ultrarelativistic domain

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Abstract. The Rutherford scattering of a classical point charge moving in an attractive field and obeying the Lorentz–Dirac equation is solved. The size of the spatial part of the incoming 4-velocity $(\gamma^2 - 1)^{1/2}$ takes the values 1000, 100 and 0.1, respectively. Asymptotic expansions of physical solutions are derived and used. Results are displayed and discussed. It is shown that all solutions satisfy physical expectations. A condition for treating radiation reaction as a perturbation is applied. Some earlier problems that have led to suggestions of unphysical features of the Lorentz–Dirac equation are explained on a physical basis.

1. Introduction

The Lorentz–Dirac (LD) equation is the equation that, in the general case, describes the motion of an elementary classical point charge (henceforth called a P-charge). It can be shown [1, 2] that this equation fits the ordinary Lorentz equation of a classical particle whose charge is distributed within a very small volume. This kind of charge is called a C-charge below.

The LD equation takes the form

$$q^2\dot{a}^\mu = 1.5Ma^\mu - 1.5qF^{\mu\nu}_{\text{ext}}u_\nu - q^2(a^\alpha a_\alpha)u^\mu\,. \tag{1}$$

In this paper Greek indices range from 0 to 3. The speed of light takes the value $c = 1$. The metric is diagonal and its entries are $(1, -1, -1, -1)$. Derivatives are carried out with respect to the particle's proper time τ. Thus, the 4-velocity is $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, the 4-acceleration is $a^\mu = \mathrm{d}u^\mu/\mathrm{d}\tau$ and its 4-derivative is $\dot{a}^\mu = \mathrm{d}a^\mu/\mathrm{d}\tau$ where an upper dot denotes differentiation with respect to τ. In equation (1), M and q denote the particle's mass and charge, respectively, and $F^{\mu\nu}_{\text{ext}}$ is the tensor of the external electromagnetic fields. In the literature, the terms of (1) that contain the q^2 factor are generally called radiation reaction. The term proper speed denotes the size of the spatial components of the 4-velocity and its value is $(\gamma^2 - 1)^{1/2}$. In the present work charges move in a plane, which is taken to be the (x, y) one. Hereafter, subscripts x and y denote the appropriate components of 4-vectors. Thus, the relativistic factor γ is

$$\gamma = (1 - v_x^2 - v_y^2)^{-1/2} = (1 + u_x^2 + u_y^2)^{1/2}\,. \tag{2}$$

Assume now that charge is not quantized and one can evaluate infinitesimal quantities. Thus, taking the limit where $M \to 0$, $q \to 0$ and $q/M =$ constant, one finds that the LD equation (1) is cast into the ordinary Lorentz equation

$$Ma^\mu = qF^{\mu\nu}u_\nu\,. \tag{3}$$

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The ordinary Lorentz equation (3) differs from the LD one (1) in two points. The terms of (1) containing the q^2 factor are eliminated in (3) due to the limit process. Besides, since $q \to 0$, the fields associated with it become infinitesimal too. Hence, these field quantities can also be ignored and the fields tensor of (3) represents the entire electromagnetic fields. It can be shown [2] that if one constructs a C-charge then the LD equation (1) is obtained from the ordinary Lorentz equation (3) if the spatial dimensions of the C-charge shrink to zero. Here the overall self force, exerted by elements of the C-charge on other charge elements of it, boils down to the terms of (1) that contain the q^2 factor.

The third-order non-Newtonian nature of the LD equation is the underlying reason for many debates of its meaning. Discussions in this subject and many references can be found in books and articles [3–7]. Some authors have gone very far and suggested replacing the LD equation by a variety of second-order Newtonian equations [8–12]. However, it has been shown that the alternative equations are physically unacceptable [13–16]. This outcome provides another indication of the significance of the LD equation.

Due to its non-Newtonian third order property, the LD equation of a P-charge contains many solutions that do not correspond to a solution of the Lorentz force exerted on a C-charge [2]. It turns out that these kinds of solutions of the LD equation have unphysical properties. Hence, the problem of the LD equation is not to show that it contains unphysical solutions but that, for every appropriate set of initial conditions, one can find a physical solution.

An exception that does not satisfy this requirement was pointed out recently [17, 18]. These publications follow earlier works [19, 20]. Consider the case of a one-dimensional motion of a P-charge attracted towards the origin by a Coulomb force. In this case it is shown that the solution to the LD equation is unphysical and the P-charge eventually turns back and moves away from the origin with a velocity that approaches the speed of light (such a solution is called a runaway solution). It is clear that a solution of this kind does not fit the motion of a C-charge satisfying the ordinary Lorentz equation (3), since, as is well known, this equation conserves energy.

A remark that can be made on this issue is that it is not clear whether or not a solution of the one-dimensional motion is the limit of solutions to the two spatial dimensions problem of the LD equation in cases where the impact parameter tends to zero [2]. Obviously, it is the limit where the impact parameter tends to zero that has a physical meaning, because, in reality, there is no motion whose impact parameter vanishes identically. This point increases the interest in the small impact parameter problem of the Rutherford scattering of a P-charge obeying the LD equation.

It is evident that if the impact parameter of a Rutherford scattering is small, acceleration at the vicinity of the origin is large, entailing radiation of a large amount of energy. It follows that in these cases the incoming particle must move ultrarelativistically. The main objective of the present work is to investigate this kind of motion and to compare it to the corresponding non-relativistic one. To this end, solutions are grouped into three sets where, in the remote past, the proper speed takes the values 1000, 100 and 0.1, respectively.

It turns out that the impact parameter of the solutions of each group has a minimum which is greater than zero. Relying on energy conservation of electrodynamics and on the derivation of the LD equation of P-charges from the Lorentz equation of C-charges, one expects that an incoming charge whose impact parameter is smaller than the above-mentioned minimum is captured by the attracting field. As explained later, solutions of the LD equation reported here are solved backwards in time, using inertial motion at $\tau = \infty, r = \infty$ as initial conditions. Hence, capture phenomena are beyond the scope of the present work.

The specific problems treated and the method used for extracting the solutions are described in the second section. The third section contains a presentation of the results and a discussion of their meaning. The outcome of the present work indicates that every case of Rutherford scattering of the LD equation has a physically acceptable solution. The fourth section contains an evaluation of similarities and differences of the consequences of this work and those of an earlier article discussing the same topic [21]. Opinions ascribing unphysical features to solutions of small impact parameter are discussed. Concluding remarks are made in the last section.

2. Solution of the Lorentz–Dirac equation

The LD equation (1) is solved here in its original form and no variable transformation is utilized. Thus, the proper time τ is the independent variable. The system of units used yields $-q_1 = q_2 = M = 1$, where q_1 is the charge of the particle fixed at the origin and q_2 is the charge of the moving particle. Hence, the external electrostatic field is

$$\boldsymbol{E} = -\boldsymbol{r}/r^3 \,. \tag{4}$$

The problem consists of two coupled third-order differential equations of the 1 and 2 (namely x and y) components of (1):

$$\dot{a}_x = 1.5a_x + 1.5\gamma \frac{x}{(x^2+y^2)^{3/2}} - (a^\alpha a_\alpha)u_x \tag{5}$$

$$\dot{a}_y = 1.5a_y + 1.5\gamma \frac{y}{(x^2+y^2)^{3/2}} - (a^\alpha a_\alpha)u_y \,. \tag{6}$$

Carrying out a straightforward calculation, one finds

$$a^\alpha a_\alpha = -(a_x^2 + a_y^2 + (u_x a_y - u_y a_x)^2)/\gamma^2 \,. \tag{7}$$

An application of (2), (7) and the definitions of u^μ and a^μ, as given after (1), indicates that (5) and (6) can be written in terms of x, y and their derivatives.

Using standard substitutions, one can cast (5) and (6) into a system of six first-order differential equations. As is shown elsewhere [21–23], a solution of the LD equation that proceeds forwards in time yields unstable and unreliable results. Hence, equations (5) and (6) are solved backwards in time. However, the terms 'in' and 'out' take their usual meaning where 'in' labels quantities at $\tau = -\infty$ and 'out' labels those of $\tau = \infty$.

In this work, the axis of the proper time $\tau \in (-\infty, \infty)$ is divided into three parts: an asymptotic part of the remote past, a finite interval when the moving charge is not very far from the origin and another asymptotic part describing the motion in the distant future. The solution at the second part is obtained from a stepwise differential equation solver which is based on the predictor-corrector method. Truncated asymptotic expansions [24, 25] are used for the asymptotic regions.

It has already been pointed out that asymptotic expansions can be used for the one spatial dimension problem of the LD equation [23, 26]. Like here, the numerical method of these papers uses three regions as defined above. The solution is fitted at the two common boundary points of the intermediary interval and each of the asymptotic regions. It is shown here that this method can also be utilized for the motion in two spatial dimensions. To this end, consider the special case where the asymptotic motion is parallel to the x-axis.

As in the case of a motion in one spatial dimension, the asymptotic expansion relies on the fact that equations (5) and (6) do not depend explicitly on time. The expansion is written as a power series in x^{-1}. Thus, one writes the x-component of the 4-velocity as

$$u_x(x) = \sum_{k=0}^{\infty} d_k x^{-k} \tag{8}$$

and the y-coordinate takes the form

$$y(x) = \sum_{k=0}^{\infty} b_k x^{-k}\,. \tag{9}$$

Using the relation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}x} = u_x \frac{\mathrm{d}}{\mathrm{d}x} \tag{10}$$

as well as (2) and (7), one can cast all terms of (5) and (6) into quantities written by means of (8), (9) and their derivatives with respect to x. The most significant term of each of the expressions (8) and (9) is defined by the initial conditions. Thus d_0 is the asymptotic proper speed and b_0 is the positive or negative value of the impact parameter. The other terms of the expansion are determined recursively. The general asymptotic expansion is obtained from the special case by applying a rotation in the (x, y) plane. It follows that a general asymptotic expansion depends on three free parameters.

The actual calculation is carried out in the following way. Given final values of the proper speed and of the impact parameter, one finds the asymptotic solution in the distant future. This solution is considered valid at points which are far enough from the origin. In most cases, the asymptotic region is defined for $|x| \geqslant 20\,000$. The values of x, u_x, a_x, y, u_y and a_y obtained at the boundary of this region are used as initial conditions for the stepwise procedure that solves the problem backwards in time. This method yields a solution at the region where the moving particle is not very far from the origin. To the solution obtained in this way one fits an asymptotic solution that holds in the remote past. In this way, one solution of the LD equation is known for all values of τ. Varying u_{out}, one can find a solution where u_{in} fits the required value (namely, 1000, 100 or 0.1, respectively) with an appropriate numerical accuracy.

3. Properties of the solutions

The discussion of this section is divided into three parts. Tests of the accuracy of the solutions are described in the first subsection. Next, some general properties of the solutions are presented and discussed. The last subsection is devoted to the relations of solutions of the LD equation to those of the ordinary Lorentz equation.

3.1. The accuracy of the solution

Two accuracy tests have been carried out. The first one is the energy balance of each solution. The quantity

$$\delta_E = \gamma_{\text{in}} - \gamma_{\text{out}} - E_{\text{rad}} \tag{11}$$

is the difference between the change in the kinetic energy of the moving charge during the entire process and the total amount of energy radiated:

$$E_{\text{rad}} = -\tfrac{2}{3}\int_{-\infty}^{\infty} a^\alpha a_\alpha \gamma \,\mathrm{d}\tau\,. \tag{12}$$

This expression is integrated numerically and the integrand is obtained from (2) and (7). Obviously, a physical solution cannot be considered accurate unless δ_E is small enough.

The second test evaluates the fit of the acceleration components a_x and a_y at the boundary of the asymptotic region of the remote past. As described in the second section, the calculation starts at $\tau = \infty$ and proceeds backwards in time. At the boundary of the asymptotic region of the remote past, a stepwise solution of equations (5) and (6) consists of six quantities

$$\{x, u_x, a_x, y, u_y, a_y\}\,. \tag{13}$$

On the other hand, as concluded in the paragraph following (10), an asymptotic expansion depends on three free parameters: the asymptotic proper speed u, the impact parameter b and the angle of rotation in the (x, y) plane. To these quantities, one should add the independent variable x, by means of which specific values of (8) and (9) are determined. Thus, there are just four independent parameters that can be used for fitting the six quantities (13).

In the present work the asymptotic expansion was calculated so that x, u_x, y and u_y took the required values as given in (13). Thus, the continuity of the components of the acceleration a_x and a_y at the boundary point discussed here indicates the accuracy of the calculations.

The two kinds of test show that the accuracy of the results is not less than seven decimal points. Hence, it can be stated that the accuracy of the numerical solutions described below is satisfactory.

3.2. Some general properties of the solutions

A graphical presentation of parts of the trajectories of several solutions is seen in figures 1(*a*) to (*c*). Each figure presents solutions that differ by the impact parameter b_{in} and, consequently, by the scattering angle. In every figure, the trajectory whose scattering angle is the largest belongs to the solution which is nearest to the capture threshold. Here and elsewhere, omission of units on figures means that units of this work are used.

In each of the figures 1(*a*) to (*c*), it is seen that the smaller the impact parameter, the larger the scattering angle. A comparison of the three figures indicates that the maximal scattering angle is larger if the initial proper speed u_{in} is smaller. These features are also seen later in other figures.

The general shape of the ultrarelativistic solutions of figures 1(*a*) and (*b*) is different from that of the non-relativistic ones, depicted in figure 1(*c*). The scattering angle of the ultrarelativistic solutions is extremely small for all trajectories whose impact parameter is not very close to the capture threshold. On the other hand, in the non-relativistic case, it is seen that the trajectories bend considerably even if the impact parameter is far from the capture threshold.

A common property of the solutions is the direction of the curvature of the trajectories. It is found that the curvature is in accordance with behaviour of a particle satisfying a second-order Newtonian equation which is *attracted* towards the origin. It should be pointed out that the LD equation (1) is a third-order equation and one cannot be sure of the shape taken by trajectories of its solutions. Thus, the curvature obtained is compatible with physical expectations.

In the present work, the number of different solutions within each set varies from about 30 to 60. These numbers enable the use of smooth curves in the following figures. The rather large variation of the impact parameter b_{in} and the significance of processes that take place in cases where this quantity is small, leads to the choice of a logarithmic scale for this variable.

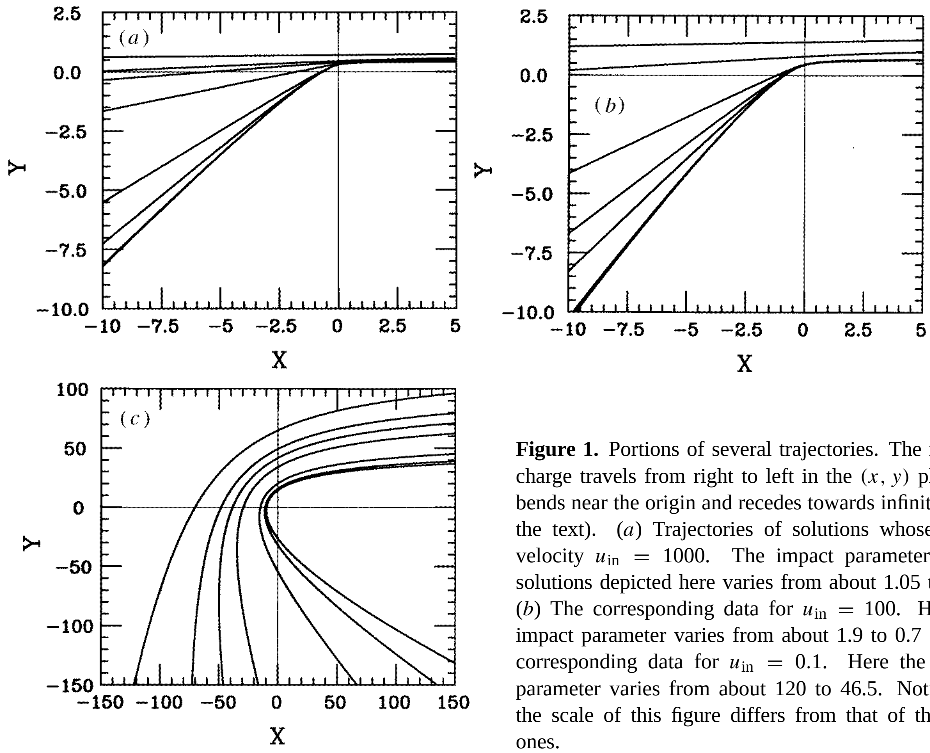

Figure 1. Portions of several trajectories. The moving charge travels from right to left in the (x, y) plane. It bends near the origin and recedes towards infinity. (See the text). (a) Trajectories of solutions whose initial velocity $u_{\rm in} = 1000$. The impact parameter $b_{\rm in}$ of solutions depicted here varies from about 1.05 to 0.43. (b) The corresponding data for $u_{\rm in} = 100$. Here the impact parameter varies from about 1.9 to 0.7 (c) The corresponding data for $u_{\rm in} = 0.1$. Here the impact parameter varies from about 120 to 46.5. Notice that the scale of this figure differs from that of the other ones.

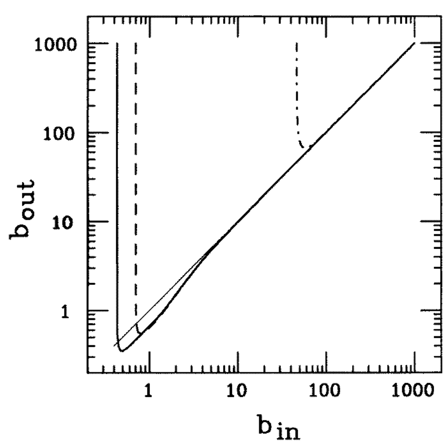

Figure 2. The impact parameter of the outgoing particle, $b_{\rm out}$, is depicted as a function of that of the incoming one $b_{\rm in}$. The full curve refers to solutions whose initial velocity $u_{\rm in} = 1000$, the broken curve pertains to solutions where $u_{\rm in} = 100$ and the chain curve represents those where $u_{\rm in} = 0.1$. The thin line satisfies $b_{\rm out} = b_{\rm in}$ and is drawn for reference.

The dependence of the impact parameter of the outgoing charge, $b_{\rm out}$, on that of the incoming one, $b_{\rm in}$, is displayed in figure 2. It is seen that for large values of $b_{\rm in}$, the relation $b_{\rm out} = b_{\rm in}$ holds to a very good approximation. On the other hand, $b_{\rm out}$ increases steeply as $b_{\rm in}$ approaches the capture threshold. These properties will be discussed in more detail in the next subsection. Examining the ultrarelativistic sets at the intermediary region, one finds that $b_{\rm out}$ becomes less than $b_{\rm in}$. This feature does not exist in the non-relativistic case $u_{\rm in} = 0.1$. One can also observe that the curves of the two ultrarelativistic sets of solutions nearly coincide if the impact parameter is not very close to the corresponding capture threshold.

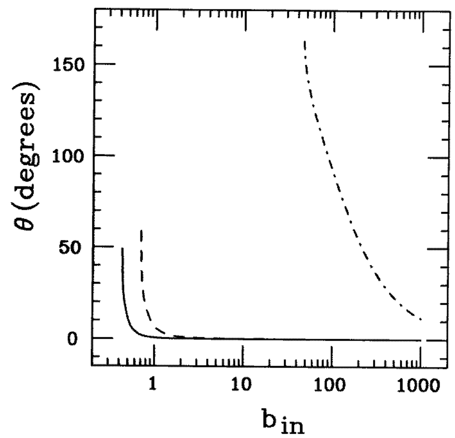

Figure 3. The dependence of the scattering angle θ on the impact parameter of the incoming charge. Each curve denotes a set of solutions as in figure 2.

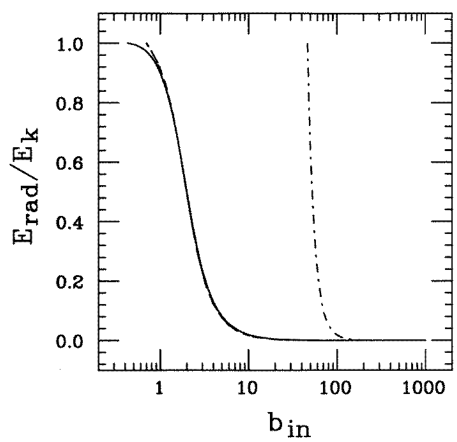

Figure 4. The ratio between the entire energy radiated and the initial kinetic energy is plotted as a function of the impact parameter of the incoming charge. Each curve denotes a set of solutions as in figure 2.

In figure 3, the scattering angle is plotted as a function of b_{in}. Referring to the ultrarelativistic cases, it is seen that for $u_{\mathrm{in}} = 1000$ the scattering angle practically vanishes if $b_{\mathrm{in}} > 1$. For $u_{\mathrm{in}} = 100$, the corresponding condition is $b_{\mathrm{in}} > 2$. On the other hand, the scattering angle of the non-relativistic case does not vanish even for $b_{\mathrm{in}} = 1000$. In all cases, the scattering angle increases as the impact parameter decreases, a property which is in accordance with intuition. A sharp increase of the scattering angle is seen near the capture threshold. This feature makes sense. Indeed, near the capture threshold, the velocity of the outgoing particle becomes smaller and the duration of its interaction with the external field is longer.

The energy radiated during the entire process is depicted in figure 4. In order to put the results of the three sets on the same scale, the ratio between the radiated energy and the initial kinetic energy is plotted. It is seen that in all cases, the amount of energy radiated increases as b_{in} decreases. This property is understandable, since approaching the origin's vicinity with a smaller impact parameter, the charge is affected by a stronger field. It makes sense to find that a motion in a stronger field, with which a bigger acceleration is expected to be involved, yields a greater amount of radiated energy. Another feature seen in this figure is the approximate coincidence of the lines of the two ultrarelativistic sets until data of $u_{\mathrm{in}} = 100$ is very close to the capture threshold. This phenomenon was also seen in figure 2.

A comparison of the data of the ultrarelativistic sets of figures 3 and 4 indicates that a small but non-negligible amount of energy is radiated for $2 < b_{\text{in}} < 10$, whereas the corresponding scattering angle practically vanishes. This property makes sense, because an ultrarelativistic particle that loses a fraction of its energy still moves ultrarelativistically. In these circumstances, its path does not bend under the influence of the not very strong external field.

3.3. The relevance of the ordinary Lorentz equation

The substitution $F^{\mu\nu} \to F^{\mu\nu}_{\text{ext}}$ into the ordinary Lorentz equation (3) yields a second-order equation that consists of the first and the second terms on the right-hand side of the LD equation (1)

$$Ma^{\mu} = qF^{\mu\nu}_{\text{ext}} u_{\nu}\,. \tag{14}$$

It should be noted that this equation is unphysical because it does not account for the energy lost to radiation. Here the relations between solutions to the Rutherford scattering of the LD equation (1) and the corresponding ones of (14) are discussed.

Obviously, the behaviour of solutions to these equations cannot be similar unless initial conditions are the same and if the radiation reaction terms of (1) can be ignored or, at least, can be treated as a small perturbation. Hereafter, this requirement is called the perturbation requirement. In the following lines it is shown how far from the origin one must go in order to satisfy the perturbation requirement.

Assume that the perturbation requirement holds. It follows that (14) is satisfied to a good approximation. The calculation is carried out for an asymptotic motion which is parallel to the x-axis and for $x > 0$. Hence $|y| \ll x$. For this reason and in order to simplify the presentation, the y-coordinate is omitted from the following expressions and no subscript is used for notation. Using equations (2) and (14), one finds that in a Coulomb field $E = -1/x^2$, the absolute value of the derivative of a is

$$|\dot{a}| = \left|\frac{\mathrm{d}}{\mathrm{d}\tau}a\right| \simeq \left|\frac{q}{M}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\gamma}{x^2}\right| \simeq \left|a\frac{2qu}{Mx}\right|. \tag{15}$$

The derivation of (15) is obtained on the basis of the relations $\gamma > 1$, $x \gg 1$ and on the assumption that (14) is a good approximation. As can be seen from the expansion (8), at asymptotic regions, the quantity $a^{\alpha}a_{\alpha}$ is smaller than $\dot{a}$ by an order of magnitude. Introducing the factor $\frac{2}{3}$, one finds from (15) that, at asymptotic regions, a necessary condition for the applicability of the perturbation requirement is

$$|4qu/3Mx| \ll 1\,. \tag{16}$$

Relation (16) proves that the perturbation requirement depends not only on the distance from the origin but also on the proper speed of the moving charge. For an ultrarelativistic motion, where $u \gg 1$, equation (16) is valid only at places that are very far from the origin. Further aspects of the perturbation requirement are discussed in the next section.

Using vectorial notation, one can derive expressions analogous to (16) which hold not only at asymptotic regions. Now, if a charge moves non-relativistically and its acceleration is small, then the perturbation requirement is expected to hold. Two aspects of this kind of motion are discussed in the rest of this section.

Let us examine the set of solutions where the initial velocity is non-relativistic. Figure 4 shows that in this case the radiated energy can be ignored for solutions where $b_{\text{in}} > 200$. Moreover, figure 2 indicates that at this region $b_{\text{out}} = b_{\text{in}}$. These properties are compatible

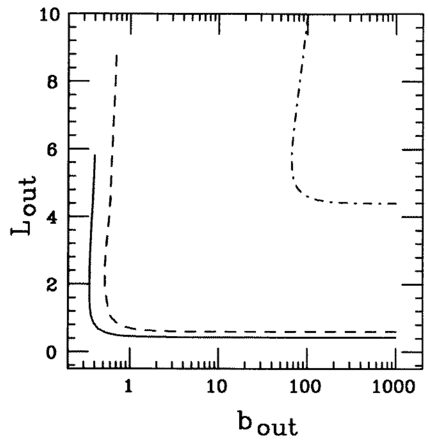

Figure 5. The final angular momentum of solutions whose b_{in} is near the capture threshold, is plotted as a function of the impact parameter b_{out} of the outgoing particle (see the text). Each curve denotes a set of solutions as in figure 2.

with those of the radiationless equation (14). This outcome is consistent with physical expectations.

A non-relativistic motion is also found along a portion of trajectories of solutions whose impact parameter b_{in} is close to the capture threshold. In these cases, the particle radiates most of its kinetic energy and recedes towards infinity in a rather slow motion. The very close approximation to a vertical line which is seen in this part of figure 2 indicates that the impact parameter b_{in} is about the same for solutions at this region. An examination of the solutions shows that the corresponding trajectories are very close until the moving charge leaves the vicinity of the origin and reaches a point P which is rather far from it. In the rest of this subsection, discussion is limited to solutions that are very near to the capture threshold. Now, if after reaching point P while trajectories are still very close to one another, the motion behaves like (14), then the variation of the particle's angular momentum is negligible.

Plots of the final angular momentum are seen in figure 5. Notice that here the horizontal axis is the vertical axis of figure 2. Hence, for each set, the solution that is nearest to the capture threshold is at $b_{out} = 1000$. In figure 5 one can see that for all the sets of solutions reported here, the final angular momentum becomes constant near the capture threshold. The foregoing discussion shows that this result is in accordance with a non-relativistic motion for which (14) is a good approximation. Now, the final angular momentum of the outgoing particle is $u_{out}b_{out}$. As a solution approaches the capture threshold, its final proper speed u_{out} decreases to zero. It follows that the impact parameter b_{out} should increase beyond all bounds as b_{in} approaches the capture threshold. This result explains the very steep feature of the plots depicted in figure 2 at the region of small b_{in}.

4. A further discussion of the physical meaning of the solutions

As reported in the previous section, *all* solutions obtained above are compatible with physical expectations. The topic of the present work has also been discussed earlier [21]. Some of the solutions reported here, where the initial proper speed is 1000, are related to those discussed in section 4 of [21]. A minor difference is associated with the asymptotic expansion used here. This expansion yields solutions defined at all values of the proper time τ and help showing the high numerical accuracy obtained here. The asymptotic expansion is not applied in [21], where approximations are used instead. Thus, in [21], the motion is assumed to be inertial at the region $r_{\mathrm{out}} > 1000$. Physically meaningful solutions have been found in [21] too. However, the authors of [21] claim to find exceptions and state that 'the peculiar behaviour noted here seems to be indicating a breaking down of the Lorentz–Dirac equation even for values of b_I somewhat larger than unity'. (See section 5 of [21]. In [21] b_I is the same as b_{in} of the present work.) It is shown in this section that the results obtained here amend this point of [21].

In order to explain the reasons for their suggestions of unphysical features of solutions having a small value of b_{in}, the authors of [21] examine the distance from the origin where energy balance is restored. A second quantity used for this purpose is the sign of the scalar product of the ordinary 3-vectors $\boldsymbol{a}\cdot\boldsymbol{v}$. A third quantity is the ratio R, which, in the notation of the present work, takes the form

$$R = a/|\gamma \boldsymbol{E}|\,. \tag{17}$$

Here $a = (a_x^2 + a_y^2)^{1/2}$, where a_x and a_y are the components of the 4-acceleration, respectively, and $\boldsymbol{E}$ is the electric field.

It is stated in section 5 of [21] that energy imbalance is found at a distance from the origin $r = 1000$ and that it indicates a peculiar behaviour of the LD equation. However, this property is completely understandable. Indeed, the contribution of the $2\dot{a}/3$ term (sometimes called the Schott term), affects the energy balance of the LD equation. As is shown above, the Schott term cannot be ignored if the perturbation requirement (16) is not satisfied. Now, at $r = 1000$, the left-hand side of (16) (which also depends on the proper speed) is greater than 1 if the proper speed $u_{\mathrm{in}} \simeq 1000$. Hence, the perturbation requirement is strongly violated. This outcome proves that energy balance should be tested at points that are much further away from the origin. It can be concluded that the claim made in [21], concerning energy imbalance at a distance $r = 1000$ cannot be used for discrediting the physical merits of the LD equation.

The change of sign of $\boldsymbol{a}\cdot\boldsymbol{v}$ of the incoming charge at large distances from the origin, indicates that, at these regions, the LD equation departs from features of the second-order equation (14), where radiation reaction is ignored. The same is true for the deviation of the ratio (17) from unity. As is well known, at large distances from the origin, the Coulomb field is weak. Now, the problem is whether or not a weak external field should be considered as a sufficient condition for the validity of the assumption that radiation reaction should be much smaller than other terms of the LD equation.

As pointed out in the introduction, the LD equation of P-charges can be derived from the Lorentz equation of C-charges [2]. To this end, the entire electromagnetic fields tensor at the location of a C-charge is split as follows:

$$F^{\mu\nu} = F^{\mu\nu}_{\mathrm{ext}} + F^{\mu\nu}_{\mathrm{self}}\,. \tag{18}$$

Here $F^{\mu\nu}_{\mathrm{ext}}$ is associated with all charges, except the C-charge whose motion is examined. The analysis shows that the effect of $F^{\mu\nu}_{\mathrm{self}}$ on the motion of a C-charge, whose spatial size shrinks to zero, can be replaced by the radiation reaction terms of the LD equation. This analysis casts a new light on the radiation reaction terms of the LD equation. In particular, if one examines the motion of C-charges, one finds no physical basis for the claim that $F^{\mu\nu}_{\mathrm{self}}$ must be much smaller than $F^{\mu\nu}_{\mathrm{ext}}$ at asymptotic regions where the latter is weak. (The two field quantities on the right-hand side of (18) have equal rights!) Turning now to P-charges, one finds that a weak $F^{\mu\nu}_{\mathrm{ext}}$ does not justify the demand that radiation reaction terms of the LD equation should make a small perturbation. As is explained above, in asymptotic regions, relation (16) is the right criterion for treating radiation reaction as a small perturbation. This relation also depends on the proper speed of the moving charge. It follows that this discussion relieves the LD equation from claims [21] of having unphysical properties in regions where $F^{\mu\nu}_{\rm ext}$ is weak.

The following analysis of the ultrarelativistic cases solved in this work show that the numerical results are consistent with the perturbation requirement (16). Let us examine each of the solutions of the two ultrarelativistic sets and find $\bar{r}$. $\bar{r}$ is the distance from the origin of the incoming charge which is taken at a point where the ratio

$$\bar{R} \equiv \tfrac{2}{3}|\dot{a}|/|a| = 0.01 \tag{19}$$

holds. At this point radiation reaction terms are still small but cannot be ignored. Evidently, the size of $\bar{r}$ determines that of the electric Coulomb field. This test indicates the rate of the relative growth of $\dot{a}$. Consequently, one finds information on deviation of the LD equation from the radiationless equation (14).

For all cases of the set where $u_{\rm in} = 100$, $\bar{r}$ varies within the small interval $[13\,372, 13\,401]$. On the other hand, the impact parameter $b_{\rm in}$ takes its full range $[1000, 0.6986]$. The following argument emphasizes this point. Evidently, $\max(b_{\rm in}) = 1000$ is much greater than $\delta\bar{r} = 29$. Moreover, if solutions whose $b_{\rm in} > 100$ are excluded then the range of $\bar{r}$ decreases considerably and one finds $\bar{r} \in [13\,400.7, 13\,401]$. The results of the set where $u_{\rm in} = 1000$ are analogous. Here $\bar{r} \in [134007, 134010]$ and $b_{\rm in} \in [1000, 0.4386]$.

The foregoing findings indicate several properties of solutions of ultrarelativistic motion at asymptotic regions. It is seen that a small, but not negligible, contribution of radiation reaction terms takes place at a very long distance from the source of the field. It is also seen that, at asymptotic regions, the relative size of the radiation reaction terms depends on the ultrarelativistic proper speed and on the distance from the origin r (namely, on the electric field), but is practically independent of the impact parameter $b_{\rm in}$. Moreover, a comparison of the results of the two ultrarelativistic sets shows that as $u_{\rm in}$ is multiplied by 10, the distance where $\bar{R} = 0.01$ is multiplied by about the same factor. This property is consistent with the form of the condition for the perturbation requirement (16) which is practically a function of (u/r).

This analysis shows that for an ultrarelativistic incoming charge, $\dot{a}$ grows faster as $u_{\rm in}$ is greater. Evidently, the influence of $\dot{a}$ may change the sign of the acceleration even at remote regions where both a and $\dot{a}$ are very small. By the same token, the ratio R of (17) might take large values while both a and $\boldsymbol{E}$ are very small. In this way, the other two arguments made in section 5 of [21], namely the change of sign of $\boldsymbol{a} \cdot \boldsymbol{v}$ and the large value of the ratio R of (17) are settled. Moreover, it is also shown that, unlike the interpretation of [21], these phenomena are practically independent of the impact parameter $b_{\rm in}$ (provided the other initial value, $u_{\rm in}$, is held fixed). The last conclusion is consistent with the mathematical structure of the equation. Indeed, at asymptotic regions, where the impact parameter $b_{\rm in} \ll r$, the LD equation is not affected by a variation of $b_{\rm in}$.

5. Concluding remarks

The LD equation is discussed in the present work. Solutions of Rutherford scattering in an attractive Coulomb field of charges whose initial proper speed takes the values 1000, 100 and 0.1, respectively, are presented. The impact parameter of the incoming particle varies from a value which is very close to the capture threshold up to 1000. It is shown that physically meaningful solutions exist for all cases discussed here. Asymptotic expansions of solutions to this equation are derived. The general shape of the trajectories is like that of a particle obeying a second-order Newtonian equation which is attracted towards the origin. A condition is found for the similarity of solutions of the LD equation and the corresponding ones of the ordinary Lorentz equation of a charge moving in a field $\boldsymbol{E} = -\boldsymbol{r}/r^3$. It is proved that at asymptotic regions, this condition depends on the distance from the origin and on the proper speed. In the non-relativistic case discussed here for $u_{\text{in}} = 0.1$, it is shown that if the impact parameter is greater than 200, then solutions of the LD equation and of the Lorentz one have the same properties. It is proved that as b_{in} approaches the capture threshold, b_{out} increases beyond all bounds. Some other general features of the solutions are displayed and discussed.

An analysis of the ultrarelativistic cases shows that radiation reaction terms become relatively significant at regions that are far from the origin. It is also shown that asymptotic phenomena are practically independent of the size of the impact parameter b_{in}. The analysis settles problems of asymptotic motion and substantiates the physical merits of the LD equation of Rutherford scattering.

It has been pointed out that there is no physical solution to the one-dimensional motion of a charge obeying the LD equation, if the Coulomb field is attractive [17–20]. The present work sets an upper limit to this problem and shows that solutions to the corresponding Rutherford scattering have a physical meaning. As a matter of fact, cases discussed in this work do not approach the limit of one-dimensional motion. Indeed, the numerical algorithm used here starts at $\tau = \infty$ when the moving charge is infinitely far from the origin and proceeds backwards in time. Therefore, the solutions found here have an impact parameter b_{in} which is greater than the capture threshold. The finite interval of impact parameters whose size is smaller than that of the capture threshold makes a gap that cannot be crossed by the numerical method used here. An attempt to investigate small impact parameters that approach the one-dimensional limit should use a different numerical algorithm. This topic is beyond the scope of the present work.

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